REPRESENTING HOMOLOGY CLASSES BY SYMPLECTIC SURFACES

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ABSTRACT. We derive an obstruction to representing a homology class of a symplectic 4-manifold by an embedded, possibly disconnected, symplectic surface.

A natural question concerning symplectic 4-manifolds is the following: Given a closed symplectic 4-manifold (M,ω) and a homology class $B \in H_2(M;\mathbb{Z})$, determine whether there exists an embedded, possibly disconnected, closed symplectic surface representing the class B. This question has been studied by H.-V. Lê and T.-J. Li [8, 9]. We always assume that the orientation of a symplectic surface is the one induced by the symplectic form. One necessary condition is then, of course, that the symplectic class $[\omega]$ evaluates positively on the class B, meaning that $\langle[\omega],B\rangle > 0$. Among other things, it is shown in [9] that a class B with $\langle[\omega],B\rangle > 0$ in a symplectic 4-manifold is always represented by a symplectic *immersion* of a connected surface. It is also noted that an obstruction to representing a homology class B by an embedded *connected* symplectic surface comes from the adjunction formula: The (even) integer

$$K_M B + B^2,$$

where K_M denotes the canonical class of the symplectic 4-manifold (M,ω), has to be at least -2. This obstruction, however, disappears, if the number of components of the symplectic surface is allowed to grow large. Note that there are examples of classes in symplectic 4-manifolds which are represented by an embedded disconnected symplectic surface, but not by a connected symplectic surface: For example in the twofold blow-up $X\#2\overline{\mathbb{CP}}^2$ of any closed symplectic 4-manifold X the sum of the classes of the exceptional spheres is not represented by a connected embedded symplectic surface according to the adjunction formula. It is the purpose of this article to derive an obstruction to representing a homology class by an embedded, possibly disconnected, symplectic surface.

In [9] it is also shown that for symplectic manifolds M of dimension at least six, every class in $H_2(M;\mathbb{Z})$ on which the symplectic class evaluates positively is represented by a connected embedded symplectic surface. In [8] there is a conjecture which in the case of symplectic 4-manifolds M says that if α is a class in $H_2(M;\mathbb{Z})$ on which the symplectic class evaluates positively, then there exists a

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positive integer N depending on α such that $N\alpha$ is represented by an embedded, not necessarily connected, symplectic surface. In the examples at the end of this article we give counterexamples to this conjecture in the 4-dimensional case.

The non-existence of an embedded symplectic surface in the class B has the following consequence for the Seiberg-Witten invariants, which we only state in the case $b_2^+ > 1$.

Proposition 1. *Let (M,ω) be a closed symplectic 4-manifold with $b_2^+(M) > 1$ and $B \neq 0$ an integral second homology class which cannot be represented by an embedded, possibly disconnected, symplectic surface. Then the Seiberg-Witten invariant of the $Spin^c$-structure*

$$s_0 \otimes PD(B)$$

is zero, where s_0 denotes the canonical $Spin^c$-structure with determinant line bundle K_M^{-1} induced by a compatible almost complex structure.

Here PD denotes the Poincaré dual of a homology class. Note that the first Chern class of the $Spin^c$-structure $s_0 \otimes PD(B)$ is equal to $-K_M + 2PD(B)$. Proposition 1 is a consequence of a theorem of Taubes, relating classes with non-zero Seiberg-Witten invariants to embedded symplectic surfaces [14].

In the following, let (M,ω) denote a closed symplectic 4-manifold and $\Sigma \subset M$ an embedded, possibly disconnected, closed symplectic surface representing a class $B \in H_2(M;\mathbb{Z})$. We always assume that the orientation of M is given by the symplectic form ($\omega \wedge \omega > 0$). If the class B is divisible by an integer $d > 1$, in the sense that there exists a class $A \in H_2(M;\mathbb{Z})$ such that $B = dA$, then there exists a d-fold cyclic ramified covering $\phi\colon \overline{M} \to M$, branched along Σ. The branched covering is again a closed symplectic 4-manifold. This is a well-known fact (the pullback of the symplectic form ω plus t times a Thom form for the preimage $\overline{\Sigma}$ of the branch locus is for small positive t a symplectic form on $\overline{M}$; see [3, 11] for a careful discussion). The invariants of $\overline{M}$ are given by the following formulas [4, p. 243], [5]:

$$\begin{aligned}
K_{\overline{M}} &= \phi^*(K_M + (d-1)PD(A)) \\
K_{\overline{M}}^2 &= d(K_M + (d-1)PD(A))^2 \\
w_2(\overline{M}) &= \phi^*(w_2(M) + (d-1)PD(A)_2) \\
\sigma(\overline{M}) &= d\left(\sigma(M) - \frac{d^2-1}{3}A^2\right)
\end{aligned}$$

Here $PD(A)_2 \in H^2(M;\mathbb{Z}_2)$ is the mod 2 reduction of $PD(A)$. The second equation follows from the first because the branched covering map has degree d.

Suppose that the branched covering $\overline{M}$ is symplectically minimal and not a ruled surface over a curve of genus greater than 1. Then theorems of C. H. Taubes and A.-K. Liu [10, 13] imply that $K_{\overline{M}}^2 \geq 0$. With the formula above, we get the following obstruction on the class A.

Theorem 2. *Let (M, ω) be a closed symplectic 4-manifold, $\Sigma \subset M$ an embedded, possibly disconnected, closed symplectic surface and $d > 1$ an integer such that $dA = [\Sigma]$ for a class $A \in H_2(M; \mathbb{Z})$. Consider the d-fold cyclic branched cover $\overline{M}$, branched along Σ. If $\overline{M}$ is minimal and not a ruled surface over a curve of genus greater than 1, then*

$$(K_M + (d-1)PD(A))^2 \geq 0.$$

It is therefore important to ensure that the branched covering $\overline{M}$ is minimal and not a ruled surface. First, we have the following lemma.

Lemma 3. *Let $\phi\colon \overline{M} \to M$ be a cyclic d-fold branched covering of closed oriented 4-manifolds. Then $b_2^+(\overline{M}) \geq b_2^+(M)$.*

Proof. With our choice of orientations, the map $\phi\colon \overline{M} \to M$ has positive degree. By Poincaré duality, the induced map $\phi^*\colon H^*(M; \mathbb{R}) \to H^*(\overline{M}; \mathbb{R})$ is injective. It maps classes in the second cohomology of positive square to classes of positive square. This implies the claim. $\square$

Proposition 4. *In the notation of Theorem 2, each of the following two conditions imply that $\overline{M}$ is minimal and has $b_2^+(\overline{M}) > 1$ and hence is not a ruled surface:*

(a) *If d is odd assume that M is spin and if d is even assume that $PD(A)$ is characteristic. Also assume that $3\sigma(M) \neq (d^2-1)A^2$.*

(b) *Assume that $b_2^+(M) \geq 2$ and there exists an integer $k \geq 2$ such that the class*
$$K_M + (d-1)PD(A)$$
is divisible by k.

Proof. Consider the d-fold branched covering $\overline{M}$, branched along Σ. The assumptions in case (a) imply that $\overline{M}$ is spin and that the signature $\sigma(\overline{M})$ is non-zero. According to a theorem of M. Furuta [2] we have $b_2^+(\overline{M}) \geq 3$. Also the symplectic manifold $\overline{M}$ is minimal, because it is spin. In case (b) the lemma implies that $b_2^+(\overline{M}) \geq 2$. In addition, the symplectic manifold $\overline{M}$ is minimal, because its canonical class is divisible by k (a non-minimal symplectic 4-manifold Y contains a symplectic sphere S with $K_Y S = -1$). $\square$

Example 5. Consider $M = K3$. Then we have $K_M = 0$. Let $d \geq 3$ be an integer and $A \in H_2(M; \mathbb{Z})$ a class with $A^2 < 0$. Theorem 2 together with Proposition 4 part (b) imply that dA is not represented by an embedded symplectic surface. Note that $K3$ contains indivisible classes of negative self-intersection which, for a suitable choice of symplectic structure, are represented by symplectic surfaces, for example symplectic (-2)-spheres. Let A be the homology class of such a sphere and $\alpha = 3A$. Then α is a counterexample to Lê's Conjecture 1.4 in [8].

Example 6. Let X be a closed symplectic spin 4-manifold with $b_2^+ > 1$ and M the blow-up $X \# \overline{\mathbb{CP}}^2$. Let E denote the class of the exceptional sphere in M. We have $K_M = K_X + PD(E)$. For every positive even integer d with $d^2 > K_X^2$, the class dE is not represented by a symplectic surface. Taking for example the blow-up of the $K3$ surface and $\alpha = 2E$, we get another counterexample to Lê's conjecture.

Note that with this method it is impossible to find a counterexample to Lê's conjecture under the additional assumption that $\alpha^2 > 0$.

In light of the second example, the following conjecture seems natural.

Conjecture. *Let M be the blow-up $X\#\overline{\mathbb{CP}}^2$ of a closed symplectic 4-manifold X and E the class of the exceptional sphere. Then dE is not represented by an embedded symplectic surface for all integers $d \geq 2$.*

This conjecture holds by a similar argument as above for X the $K3$ surface and the 4-torus T^4. Moreover, using positivity of intersections, the conjecture holds in the complex category for the blow-up of a complex surface and embedded complex curves. In fact, in this category it generalizes to multiples of the class of any connected embedded complex curve with negative self-intersection.

Remark 7. Branched covering arguments have been used in the past to find lower bounds on the genus of a connected surface representing a divisible homology class in a closed 4-manifold, see [1, 6, 7, 12].

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